

A QUASI-OPTIMAL ERROR ESTIMATE FOR A DISCRETE SINGULARLY PERTURBED APPROXIMATION TO THE PRESCRIBED CURVATURE PROBLEM

MAURIZIO PAOLINI

ABSTRACT. Solutions of the so-called *prescribed curvature problem* $\min_{A \subseteq \Omega} \mathcal{P}_\Omega(A) - \int_A g(x)$, g being the curvature field, are approximated via a singularly perturbed elliptic PDE of bistable type. For *nondegenerate* relative minimizers $A \subset\subset \Omega$ we prove an $\mathcal{O}(\epsilon^2 |\log \epsilon|^2)$ error estimate (where ϵ stands for the perturbation parameter), and show that this estimate is quasi-optimal. The proof is based on the construction of accurate barriers suggested by formal asymptotics. This analysis is next extended to a finite element discretization of the PDE to prove the same error estimate for discrete minima.

1. INTRODUCTION

Given an open bounded domain Ω in \mathbf{R}^n with Lipschitz continuous boundary and a curvature field $g \in L^\infty(\Omega)$, we address the solution of the so-called *prescribed curvature problem*, which consists of minimizing the functional

$$(1.1) \quad \tilde{\mathcal{G}}(A) = \mathcal{P}_\Omega(A) - \int_A g(x) \, dx, \quad A \subseteq \Omega.$$

Here, $\mathcal{P}_\Omega(A)$ denotes the perimeter in Ω of the Caccioppoli set $A \subseteq \Omega$, i.e., the $(n-1)$ -dimensional Hausdorff measure of $\partial^* A \cap \Omega$, where $\partial^* A$ is the reduced boundary and coincides with ∂A for regular sets [17]. Problems involving a surface energy together with a volume term arise in many fields, e.g. in phase transition problems, capillarity, minimal surfaces [1], [18], [27].

Any absolute or relative minimizer A to $\tilde{\mathcal{G}}$ is known to verify the condition that the sum of the principal curvatures at any point $x \in \partial A \cap \Omega$ equals $g(x)$ (wherever g is continuous) and that the interface ∂A meets the boundary $\partial \Omega$ orthogonally.

Given $\epsilon > 0$, the functional $\tilde{\mathcal{G}}$ in (1.1) can be approximated by the more regular *relaxed functional* [25]

$$(1.2) \quad \mathcal{F}_\epsilon(v) = \epsilon \int_\Omega |\nabla v|^2 \, dx + \frac{1}{\epsilon} \int_\Omega \Psi(v) \, dx - \frac{c_0}{2} \int_\Omega gv \, dx, \quad \forall v \in H^1(\Omega),$$

in the sense of De Giorgi's Γ -convergence [13] as $\epsilon \rightarrow 0$. Here, $\Psi(t) = (1 - t^2)^2$ is a double-well potential which penalizes values of v other than ± 1 and $c_0 =$

Received by the editor September 28, 1994 and, in revised form, August 9, 1995.

1991 *Mathematics Subject Classification*. Primary 35B25, 35J60, 65N30; Secondary 35A35, 49Q05 .

This work was partially supported by MURST (Progetto Nazionale "Equazioni di Evoluzione e Applicazioni Fisco-Matematiche" and "Analisi Numerica e Matematica Computazionale") and CNR (IAN and Contracts 92.00833.01, 93.00564.01) of Italy.

$\int_{-1}^1 \sqrt{\Psi(s)} ds$. It follows that any sequence of absolute minimizers u_ϵ of \mathcal{F}_ϵ converges in $L^1(\Omega)$, up to a subsequence, to a function $u \in BV(\Omega; \{-1, 1\})$ such that the set $A = \{x \in \Omega : u(x) = 1\}$ is an absolute minimizer of $\tilde{\mathcal{G}}$.

The first result of this paper is an essentially $\mathcal{O}(\epsilon^2)$ quasi-optimal interface error estimate for minimizers of \mathcal{F}_ϵ that reads as follows. Let $A \subset\subset \Omega$ be an absolute or relative minimizer of $\tilde{\mathcal{G}}$ with strictly positive principal eigenvalue λ_1 of the second variation of $\tilde{\mathcal{G}}$ (A is a *nondegenerate* minimizer). Then there exist a relative minimizer u_ϵ of \mathcal{F}_ϵ , a positive constant C depending on Ω, g, λ_1 , and a threshold value ϵ_0 (possibly depending also on A and C) such that, setting $\Sigma_\epsilon = \{x \in \Omega : u_\epsilon(x) = 0\}$, one has

$$\text{dist}_H(\partial A, \Sigma_\epsilon) \leq C\epsilon^2 |\log \epsilon|^2, \quad \forall \epsilon \leq \epsilon_0.$$

Here, dist_H denotes the Hausdorff distance between sets.

The Euler-Lagrange equation associated with \mathcal{F}_ϵ turns out to be a singularly perturbed nonlinear elliptic PDE, which can be solved by finite elements techniques [4], [5]. We then prove that the $\mathcal{O}(\epsilon^2 |\log \epsilon|^2)$ interface error estimate remains valid also for solutions of these discrete versions of the elliptic PDE, provided the mesh size h decreases with ϵ at least as $h = \mathcal{O}(\epsilon^{5/2})$. The discrete problem introduced here actually differs from those presented in [4], [5] in the absence of mass lumping in the potential term. Mass lumping in that term can however be taken into account at the cost of a much stronger requirement on the size of h with respect to ϵ . Results in the same spirit have been recently obtained by Dziuk and Hutchinson [15] for a discrete version of the Plateau problem using a front-tracking approach.

Solutions of the prescribed curvature problem can also be obtained by considering the stationary limit as $t \rightarrow +\infty$ of the evolutionary reaction-diffusion equation associated with the Euler-Lagrange equation,

$$(1.3) \quad u_t - \Delta u + \frac{1}{2\epsilon^2} \Psi'(u) = \frac{c_0}{2\epsilon} g.$$

This so-called Allen-Cahn equation in fact approximates, by its zero level-set, the evolution of a surface moving with normal velocity given by the local mean curvature plus the forcing term g with an error of order $\mathcal{O}(\epsilon^2 |\log \epsilon|^2)$ [3], [28], [29], [30], [31], [32], [33]. Equation (1.3) has been thoroughly investigated theoretically in the literature; see, e.g., [1], [2], [7], [8], [9], [12], [14], [16], [23] and the references cited therein.

The outline of the paper is as follows. In §§2 and 3 we fix some notation and introduce the *prescribed curvature problem*. The first and second variation of the energy functional are presented in §4 in order to introduce the concept of *nondegeneracy* of relative minimizers. The comparison lemma of §5 is the main tool to establish existence of approximating solutions. Construction of appropriate barriers, based on the formal asymptotics sketched in §6, is accomplished in §7. Based on this construction, we obtain the error estimate for the relaxed functional of §8; the optimality of this estimate is then shown in §9. A similar error estimate can be obtained for a discrete version of the energy functional, as shown in §10, following the same conceptual path as for the continuous counterpart. We finish the paper with some remarks and conclusions.

2. NOTATIONS AND ASSUMPTIONS

Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open domain with Lipschitz continuous boundary, and let $A \subset\subset \Omega$ with smooth boundary $\Sigma = \partial A$; we require that A lie locally on one side of Σ and we suppose for definiteness that A is an open subset of Ω . We require $\Sigma \in \mathcal{C}^2$ in order to have continuous principal curvatures $\kappa_1, \dots, \kappa_{n-1}$; additional regularity of Σ is implicitly required by the assumption (2.1) below.

The *signed distance* function with respect to A is defined by

$$d(x) = \text{dist}(x, \mathbf{R}^n \setminus A) - \text{dist}(x, A)$$

and satisfies $|d(x)| = \text{dist}(x, \Sigma)$.

Given $D > 0$, we define the tubular neighborhood

$$\mathcal{T} = \{x \in \Omega : |d(x)| \leq D\}.$$

In view of the \mathcal{C}^2 regularity of Σ and the compact embedding $A \subset\subset \Omega$, we can choose D so small that the signed distance d is smooth over \mathcal{T} ; it turns out that the projection $s(x)$ of any $x \in \mathcal{T}$ onto Σ , defined by

$$\text{dist}(x, s(x)) = |d(x)|,$$

is uniquely determined. Theorems 8.1 and 10.2 will be proved under the following regularity assumptions for d and g :

$$(2.1) \quad d \in W^{3,\infty}(\mathcal{T}), \quad g \in W^{3,\infty}(\Omega).$$

We recall the definition of the double-well potential $\Psi(t) = (1 - t^2)^2$, and we set $\psi(t) = \frac{1}{2}\Psi'(t)$.

3. THE PRESCRIBED CURVATURE PROBLEM

Let us consider a slightly more general form for the energy functional $\tilde{\mathcal{G}}$ obtained by the addition of a boundary term. Namely, let

$$\tilde{\mathcal{F}}(A) = \mathcal{P}_\Omega(A) - \int_A g(x) \, dx + \int_{\partial\Omega \cap \partial A} \mu(x) \, d\mathcal{H}^{n-1}.$$

Here, \mathcal{H}^{n-1} denotes the $(n-1)$ -dimensional Hausdorff measure and A is any Caccioppoli subset of Ω . Let $\mu \in L^\infty(\partial\Omega; [-1, 1])$ be a weight factor for that part of ∂A lying on $\partial\Omega$. It is easy to check that the boundary of minimizers of $\tilde{\mathcal{F}}$ meets the boundary $\partial\Omega$ at an angle given by $\arccos \mu$ [18].

However, in this paper we are only concerned with relative minimizers A compactly contained in Ω , which are clearly not affected by changes in this boundary term. We therefore fix our attention to the particular choice $\mu = -1$ and the corresponding Dirichlet boundary value -1 for \mathcal{F}_ϵ . This choice alleviates the boundary layer of minimizers of the relaxed functional, again for compactly contained minimizing sets. Indeed, far outside A such minimizers are expected to assume approximately the value $-1 + C\epsilon g$, and we have only an $\mathcal{O}(\epsilon)$ discrepancy with the imposed boundary datum. The constant C depends on the double-well potential, and is actually zero with a *nonregular* choice of Ψ [33], [29]. Note that the original functional $\tilde{\mathcal{G}}$ corresponds to the choice $\mu = 0$.

The functional $\tilde{\mathcal{F}}$ is still relaxed by the functional \mathcal{F}_ϵ defined in (1.2); the only difference lies in the choice of the appropriate Dirichlet boundary condition, which depends on μ and on the double-well potential [4], [5].

We shall restrict attention in the sequel to

$$(3.1) \quad \tilde{\mathcal{F}}(A) = \mathcal{H}^{n-1}(\partial A) - \int_A g(x) \, dx.$$

Correspondingly, we shall impose on \mathcal{F}_ϵ a Dirichlet boundary condition by defining

$$D(\mathcal{F}_\epsilon) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = -1\}$$

and minimizing \mathcal{F}_ϵ over the convex set $D(\mathcal{F}_\epsilon)$. The following Γ -convergence result is well known [25]. If \mathcal{F}_ϵ is extended to $+\infty$ outside $D(\mathcal{F}_\epsilon)$, then \mathcal{F}_ϵ Γ -converges to $\frac{c_0}{2}\mathcal{F}$ in the $L^1(\Omega)$ topology, where \mathcal{F} is defined on $BV(\Omega; \{-1, 1\})$ by

$$\mathcal{F}(v) = \int_\Omega |Dv| - \int_\Omega gv \, dx + \int_{\partial\Omega} v \, d\mathcal{H}^{n-1}$$

and is extended to $+\infty$ outside $BV(\Omega; \{-1, 1\})$. Here, $\int_\Omega |Dv|$ stands for the total variation of the BV function v [21]. In terms of relative and absolute minimizers, \mathcal{F} and $\tilde{\mathcal{F}}$ are perfectly interchangeable with the position $A \rightarrow 2\varphi_A - 1$, where φ_A is the usual characteristic function of the set A . This Γ -convergence result implies that any sequence of absolute minimizers of \mathcal{F}_ϵ converge up to a subsequence to an absolute minimizer of \mathcal{F} ; however, it does not give any information on the rate of convergence.

The relaxed functional \mathcal{F}_ϵ can be easily discretized numerically by using, e.g., conforming piecewise linear finite elements. The resulting discrete functional is shown to Γ -converge to $\frac{c_0}{2}\mathcal{F}$ [4] under the essential condition $h = o(\epsilon)$, where h denotes the mesh size. The sharp error estimate for the continuous relaxed functional \mathcal{F}_ϵ is a crucial step in proving the discrete interface error estimate of §10.

The Euler-Lagrange equation corresponding to the minimization of \mathcal{F}_ϵ over $D(\mathcal{F}_\epsilon)$ reads as follows:

$$(3.2) \quad \mathcal{L}(u_\epsilon) = 0 \quad \text{in } \Omega, \quad u_\epsilon = -1 \quad \text{on } \partial\Omega,$$

where

$$(3.3) \quad \mathcal{L}(v) = -\epsilon^2 \Delta v + \psi(v) - \epsilon \frac{c_0}{2} g.$$

The operator \mathcal{L} can be regarded as a mapping of $D(\mathcal{F}_\epsilon) \subseteq H^1(\Omega)$ into $H^{-1}(\Omega)$.

4. STABILITY OF SURFACES OF PRESCRIBED CURVATURE

We now turn our attention to the properties satisfied by a relative minimizer $A \subset\subset \Omega$ of $\tilde{\mathcal{F}}$ defined in (3.1). We shall further suppose that the boundary $\Sigma = \partial A$ is of class C^2 , so that the sum of the principal curvatures κ^m is well defined. Note that this is not a consequence of the regularity of g , at least not if the ambient space has dimension ≥ 8 , as the Simons cone [6], [36] example reveals.

We shall consider the first and the second variation $\delta\tilde{\mathcal{F}}_A$ and $\delta^2\tilde{\mathcal{F}}_A$ of $\tilde{\mathcal{F}}$ associated with normal displacements of the boundary Σ . More precisely, let n denote the outward unit vector normal to Σ . For any test function $\xi \in C^2(\Sigma)$ define

$$\begin{aligned} \delta\tilde{\mathcal{F}}_A(\xi) &= \frac{d}{d\lambda} [\tilde{\mathcal{F}}(A_{\lambda\xi})]_{\lambda=0}, \\ \delta^2\tilde{\mathcal{F}}_A(\xi, \xi) &= \frac{d^2}{d\lambda^2} [\tilde{\mathcal{F}}(A_{\lambda\xi})]_{\lambda=0}, \end{aligned}$$

where $A_{\lambda\xi}$ is the set whose boundary is obtained by moving any point $x \in \Sigma$ to the new position $x + \lambda\xi(x)n(x)$. The set $A_{\lambda\xi}$ is well defined provided $|\lambda|$ is sufficiently small. A straightforward computation gives the first variation

$$\delta\tilde{\mathcal{F}}_A(\xi) = \int_{\Sigma} (\kappa^m - g)\xi \, d\mathcal{H}^{n-1},$$

which provides the usual stationarity condition

$$(4.1) \quad \kappa^m = g \quad \text{on } \Sigma.$$

The second variation is given by

$$(4.2) \quad \delta^2\tilde{\mathcal{F}}_A(\xi, \xi) = \int_{\Sigma} [|\nabla_{\Sigma}\xi|^2 - (\nabla g \cdot n + \kappa^s)\xi^2] \, d\mathcal{H}^{n-1},$$

where κ^s is the sum of the squares of the principal curvatures, and the notation ∇_{Σ} denotes the tangential gradient of functions defined on Σ . The symmetric bilinear form on $H^1(\Sigma)$ associated with the second variation (4.2) is

$$\delta^2\tilde{\mathcal{F}}_A(\xi, \eta) = (\nabla_{\Sigma}\xi, \nabla_{\Sigma}\eta)_{\Sigma} - ((\nabla g \cdot n + \kappa^s)\xi, \eta)_{\Sigma},$$

where $(\cdot, \cdot)_{\Sigma}$ denotes the L^2 scalar product in Σ .

The minimality of A implies $\delta^2\tilde{\mathcal{F}}_A(\xi, \xi) \geq 0$ for any test function ξ . The main results of the paper will be proved under a more restrictive *nondegeneracy* condition.

Definition 4.1. A regular subset $A \subset\subset \Omega$ is a *nondegenerate* relative minimizer of $\tilde{\mathcal{F}}$ if (4.1) holds and $\delta^2\tilde{\mathcal{F}}_A(\xi, \xi) > 0$ for all $\xi \neq 0$.

Let the linear operator $L : H^2(\Sigma) \rightarrow L^2(\Sigma)$ be defined by

$$\delta^2\tilde{\mathcal{F}}_A(\xi, \eta) = (L\xi, \eta)_{\Sigma}, \quad \forall \eta \in H^1(\Sigma),$$

that is, $L\xi = -\Delta_{\Sigma}\xi - (\nabla g \cdot n + \kappa^s)\xi$, where Δ_{Σ} is the Laplace-Beltrami operator on Σ associated with the metric induced by \mathbf{R}^n .

The eigenvalues of L satisfy $\lambda_1 \leq \lambda_2 \leq \dots$ and $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$. Let $h \in H^2(\Sigma)$ be the eigenfunction associated with the principal eigenvalue λ_1 , defined by

$$(4.3) \quad -\Delta_{\Sigma}h - (\nabla g \cdot n + \kappa^s)h = \lambda_1 h$$

and satisfying [20, §8.12], [11, p.451]

$$(4.4) \quad h_{\max} \geq h(x) \geq h_{\min} > 0, \quad \forall x \in \Sigma.$$

Lemma 4.1. *A subset $A \subset\subset \Omega$ satisfying (4.1) is a nondegenerate minimizer if and only if $\lambda_1 > 0$.*

Proof. If the test function ξ is normalized so that $\|\xi\|_{L^2(\Sigma)} = \|h\|_{L^2(\Sigma)}$, then $\delta^2\tilde{\mathcal{F}}_A(\xi, \xi) \geq \delta^2\tilde{\mathcal{F}}_A(h, h) = \lambda_1(h, h)_{\Sigma} = \lambda_1\|h\|_{L^2(\Sigma)}^2 > 0$. \square

Remark 4.1. The eigenfunction $h > 0$ satisfying property (4.4) plays an essential role in the construction of the subsolution, since the horizontal shift is modelled accordingly. Instead, the choice of a space-independent shift, as done in [3], [28], [31], [29], leads to a subsolution only if $\nabla g \cdot n + \kappa^s < 0$ on Σ . Note that this is a severe restriction which, in turn, implies $\lambda_1 > 0$ in view of (4.2).

5. COMPARISON LEMMA

The main tool to establish existence of an approximating solution u_ϵ is the following comparison lemma. It is a variant of known similar results (see [37]).

Lemma 5.1. *Suppose that $v_\epsilon^- \leq v_\epsilon^+ \in W^{2,\infty}(\Omega)$ are sub- and supersolutions to (3.2), that is, satisfy the inequalities $\mathcal{L}(v_\epsilon^-) \leq 0$ and $\mathcal{L}(v_\epsilon^+) \geq 0$ a.e. in Ω , and also that $v_\epsilon^- \leq -1 \leq v_\epsilon^+$ on $\partial\Omega$. Then there exists a solution $u_\epsilon \in W^{2,\infty}(\Omega)$ of problem (3.2) satisfying $v_\epsilon^- \leq u_\epsilon \leq v_\epsilon^+$. If, in addition, there exists a positive constant C such that*

$$(5.1) \quad \mathcal{L}(v_\epsilon^-) \leq -C < 0 \quad \text{and} \quad \mathcal{L}(v_\epsilon^+) \geq C > 0,$$

then u_ϵ can be chosen to be a relative minimizer of the functional \mathcal{F}_ϵ .

Proof. Set $K = \{v \in H^1(\Omega) : v^- \leq v \leq v^+, v|_{\partial\Omega} = -1\}$. Let $u_\epsilon \in K$ be an absolute minimizer of \mathcal{F}_ϵ restricted to K (which is a nonempty closed convex set), which exists by the direct method in Calculus of Variations [24]. Then u_ϵ satisfies the following variational inequality:

$$(5.2) \quad \langle \mathcal{L}u_\epsilon, v - u_\epsilon \rangle \geq 0 \quad \forall v \in K.$$

From the regularity theory of elliptic variational inequalities it follows that $u \in W^{2,\infty}(\Omega)$ [19]. Let $x \in \Omega$ be any regular point for v_ϵ^- , v_ϵ^+ , and u_ϵ . If $v_\epsilon^-(x) < u_\epsilon(x) < v_\epsilon^+(x)$, we easily get $\mathcal{L}(u_\epsilon)(x) = 0$ by appropriately choosing the test function in (5.2). Suppose now $v_\epsilon^-(x) = u_\epsilon(x) < v_\epsilon^+(x)$. By definition of \mathcal{L} , (3.3), we have $\mathcal{L}(u_\epsilon)(x) \leq \mathcal{L}(v_\epsilon^-)(x) \leq 0$. On the other hand, we get the opposite inequality $\mathcal{L}(u_\epsilon)(x) \geq 0$ by enforcing (5.2) with an appropriate choice of the test function, and again we obtain $\mathcal{L}(u_\epsilon)(x) = 0$. The remaining cases $v_\epsilon^-(x) < u_\epsilon(x) = v_\epsilon^+(x)$, $v_\epsilon^-(x) = u_\epsilon(x) = v_\epsilon^+(x)$ can be treated similarly, and we conclude that u_ϵ is a solution of problem (3.2). If we have the stronger conditions (5.1), assumptions $v_\epsilon^-(x) = u_\epsilon(x)$ and $v_\epsilon^+(x) = u_\epsilon(x)$ lead to a contradiction, and we conclude that u_ϵ is contained in the interior of K . \square

6. FORMAL ASYMPTOTICS

A formal asymptotic analysis of the singularly perturbed PDE (3.2) suggests the validity of an optimal quadratic error estimate between the surface of prescribed curvature $\Sigma \subset\subset \Omega$ and the zero level-set of the relaxed solution u_ϵ of (3.2). This error estimate will be proved via the construction of precise barriers suggested by the formal asymptotics mentioned above, which however plays no role in the rigorous proof.

6.1. Shape functions. We preliminarily need to define the shape functions; they appear in the inner expansion, and their asymptotic properties and estimates are crucial for the derivations in §7. We shall recall them briefly and refer to [3] for details. Let the linear operator $\mathcal{A} : H_{\text{loc}}^2(\mathbf{R}) \rightarrow L_{\text{loc}}^2(\mathbf{R})$ be defined by

$$(6.1) \quad \mathcal{A}\zeta = \zeta'' - \psi'(\gamma)\zeta,$$

where γ is the unique strictly increasing solution of $\gamma'' - \psi(\gamma) = 0$ satisfying $\gamma(0) = 0$ (for the choice of the potential made in §1 we actually have $\gamma(y) = \tanh(y)$). The

shape functions are defined by the equations

$$\begin{aligned}\mathcal{A}\eta &= \gamma' - \frac{c_0}{2}, \\ \mathcal{A}\xi^{(1)} &= \eta', \\ \mathcal{A}\xi^{(2)} &= \frac{1}{2}\eta^2\psi''(\gamma), \\ \mathcal{A}\xi^{(3)} &= y\gamma'.. \end{aligned}$$

Since the right-hand sides (say f) satisfy the orthogonality condition $\int_{\mathbf{R}} f\gamma' dx = 0$, any such solution (say ζ) actually exists and is unique if we further require $\zeta(0) = 0$ and a polynomial growth at infinity [3]. They satisfy the following properties (see [3, §3.2]):

$$(6.2) \quad \begin{aligned} \lim_{y \rightarrow \pm\infty} \eta(y) &= \eta_\infty = \frac{c_0}{2\psi'(1)}, \\ \lim_{y \rightarrow \pm\infty} \xi^{(1)}(y) &= \lim_{y \rightarrow \pm\infty} \xi^{(3)}(y) = 0, \\ \lim_{y \rightarrow \pm\infty} \xi^{(2)}(y) &= \pm \xi_\infty^{(2)} = \mp \frac{c_0^2\psi''(1)}{8(\psi'(1))^3}. \end{aligned}$$

6.2. Inner expansion. We focus our attention here to the sole inner expansion, which furnishes all the important information and allows us to define the building blocks in devising the appropriate shape of the sub- and supersolution.

Let $y = d_\epsilon/\epsilon$, where d_ϵ is the signed distance to the set $\Sigma_\epsilon = \{u_\epsilon = 0\}$, positive inside. Let $\Gamma_\epsilon : S \rightarrow \Sigma_\epsilon$ be a one-to-one parametrization of Σ_ϵ , where S is a reference manifold of the same topological type as Σ_ϵ , assumed to be uniform in ϵ . We also denote by $s : \Omega \rightarrow S$ the projection of any $x \in \Omega$ onto the set Σ_ϵ , so that

$$x = \Gamma_\epsilon(s(x)) - d_\epsilon(x)n_\epsilon(\Gamma_\epsilon(s(x))),$$

where n_ϵ denotes the outward unit normal vector to Σ_ϵ . Setting $U_\epsilon(y, s) = u_\epsilon(x)$, we assume the existence of asymptotic expansions for U_ϵ , Γ_ϵ , n_ϵ , κ_ϵ^m , κ_ϵ^s up to the appropriate order in terms of ϵ :

$$\begin{aligned} U_\epsilon &= U_0 + \epsilon U_1 + \epsilon^2 U_2 + \mathcal{O}(\epsilon^3), & \Gamma_\epsilon &= \Gamma_0 + \epsilon \Gamma_1 + \mathcal{O}(\epsilon^2), \\ n_\epsilon &= n_0 + \mathcal{O}(\epsilon), & \kappa_\epsilon^m &= \kappa_0^m + \epsilon \kappa_1^m + \mathcal{O}(\epsilon^2), & \kappa_\epsilon^s &= \kappa_0^s + \mathcal{O}(\epsilon), \end{aligned}$$

and define $\Sigma_0 = \{\Gamma_0(s) : s \in S\}$. By substitution of U_ϵ in (3.2), and collecting all terms of the same order, we obtain equations for each U_i , which are solvable if and only if an appropriate orthogonality condition for the right-hand sides is satisfied (Fredholm alternative) [3], [29], [33]. In particular, it follows that $\kappa_0^m = g(\Gamma_0(s(x)))$ and $\kappa_1^m = 0$, which means that the curvature of Σ_0 must equal the forcing term g , as expected, so that we are indeed approximating a minimizer of $\tilde{\mathcal{F}}$, and that $\Gamma_1 = 0$, which means that the approximation is actually second-order accurate.

The resulting expressions for U_i , $i = 0, 1, 2$, read

$$(6.3) \quad \begin{aligned} U_0 &= \gamma(y), \\ U_1 &= g(x)\eta(y), \\ U_2 &= (g^2(x) - 2g'(s))\xi^{(1)}(y) + g^2(x)\xi^{(2)}(y) + (\kappa_0^s(s) - g'(s))\xi^{(3)}(y), \end{aligned}$$

where for convenience we use $g'(s)$ as a shorthand for $-\nabla g(\Gamma_0(s)) \cdot n_0(\Gamma_0(s))$.

Observe that the function g in the definition of U_1 is evaluated at the local position $x = \Gamma_\epsilon(s) - \epsilon \gamma n_\epsilon(\Gamma_\epsilon(s))$ (which actually depends on ϵ) rather than on $\Gamma_0(s)$. This $\mathcal{O}(\epsilon)$ change affects the subsequent definition of U_2 , but is essential to allow the subsolution constructed according to (6.3) to be extended smoothly far from the interface.

6.3. Boundary layer formal asymptotics. The barrier to be constructed in §7 has to match the Dirichlet boundary value -1 . For this purpose we need to devise an appropriate shape, which is again suggested by the first three terms of the boundary layer formal asymptotics.

The derivation is quite similar to the case of the inner expansion at the transition layer, but now the Fredholm alternative does not give any orthogonality condition to be satisfied for the right-hand side of the equations defining the shape functions, because the associated homogeneous equation has trivial kernel. It is also possible now to give explicit definitions for the above-mentioned shape functions, which we give in (7.12) of §7.3, where they are used to actually construct the lower barrier near $\partial\Omega$.

7. CONSTRUCTION OF A SUBSOLUTION

Based on the eigenfunction h constructed in §4, we define for any $\epsilon > 0$ on \mathcal{T} the *modified signed distance* as follows:

$$d_\epsilon^-(x) = d(x) - c_1 \bar{h}(x) \epsilon^2 |\log \epsilon|^2,$$

where $\bar{h}(x) = h(s(x))$ and $c_1 > 0$ is a constant to be chosen later independent of ϵ . We set

$$\mathcal{T}_\epsilon^- = \{x \in \mathcal{T} : |d_\epsilon^-(x)| \leq 2\delta\epsilon |\log \epsilon|\},$$

where $\delta \geq 3$ and note that $\mathcal{T}_\epsilon^- \subset\subset \mathcal{T}$ for ϵ sufficiently small. Since \bar{h} is constant along normal directions, we have $\nabla d \cdot \nabla \bar{h} = 0$, hence

$$(7.1) \quad |\nabla d_\epsilon^-|^2 = 1 + c_1^2 \mathcal{O}(\epsilon^4 |\log \epsilon|^4).$$

Here and throughout this section the notation $f = \mathcal{O}(\epsilon^4 |\log \epsilon|^4)$ stands for $|f| \leq C\epsilon^4 |\log \epsilon|^4$ for ϵ sufficiently small, say $\epsilon < \epsilon_0$, where $C > 0$ is some constant independent of ϵ and of the constant c_1 , as well as c_2 and c_3 to be defined later on. The value ϵ_0 , instead, might depend on c_i , $i = 1, 2, 3$. Note that $c_i o(\omega(\epsilon)) = \mathcal{O}(\omega(\epsilon))$, $i = 1, 2, 3$, where ω is any positive function of ϵ .

Finally, the stretched variable is defined on \mathcal{T} by

$$y = \frac{d_\epsilon^-(x)}{\epsilon}.$$

As in [3], with the important difference of the definition of the modified distance function $d_\epsilon^-(x)$ and the related stretched variable y , the definition of the subsolution v_ϵ^- is based on the formal asymptotics sketched in §6.

Also, the existence of a boundary layer for u_ϵ , owing to the Dirichlet boundary condition, requires appropriate consideration.

Setting $y_\epsilon = \delta |\log \epsilon|$, we need to modify the shape functions $\gamma, \eta, \xi^{(1)}, \xi^{(2)}, \xi^{(3)}$ outside some compact interval $[-y_\epsilon, y_\epsilon]$ in order to let them become constant outside the larger interval $[-2y_\epsilon, 2y_\epsilon]$. This is a necessary step in order to unambiguously

extend the definition of v_ϵ^- to the whole Ω . If ζ is any of the shape functions, denoting $\zeta_{\pm\infty} = \lim_{y \rightarrow \pm\infty} \zeta(y)$, we define

$$\zeta_\epsilon(y) = \begin{cases} \zeta(y) & \text{if } |y| < y_\epsilon, \\ P_\zeta^\pm(y) & \text{if } y_\epsilon \leq |y| \leq 2y_\epsilon, \\ \zeta_{+\infty} & \text{if } y > 2y_\epsilon, \\ \zeta_{-\infty} & \text{if } y < -2y_\epsilon, \end{cases}$$

where P_ζ^\pm are two cubic polynomials suitably chosen so as to have $\zeta_\epsilon \in C^1(\mathbf{R})$.

We are now ready to define v_ϵ^- within the tubular neighborhood \mathcal{T} as follows:

$$(7.2) \quad v_\epsilon^-(x) = \gamma_\epsilon(y) + \epsilon g(x) \eta_\epsilon(y) + \epsilon^2 \mathbf{b}(x) \cdot \xi_\epsilon(y) - c_2 \epsilon^3 |\log \epsilon|^2, \quad \forall x \in \mathcal{T},$$

where we used the compact vector notation $\xi_\epsilon = (\xi_\epsilon^{(1)}, \xi_\epsilon^{(2)}, \xi_\epsilon^{(3)})$ and $\mathbf{b} = (g^2 - 2\nabla d \cdot \nabla g, g^2, \bar{\kappa}^s - \nabla d \cdot \nabla g)$; $\bar{\kappa}^s$ is the extension of κ^s to \mathcal{T} , defined by $\bar{\kappa}^s(x) = \kappa^s(s(x))$. Here, c_2 is an appropriate constant to be chosen later.

Since the modified shape functions are constant outside \mathcal{T}_ϵ^- and in particular $\xi_{\pm\infty}^{(1)} = \xi_{\pm\infty}^{(3)} = 0$, we can extend v_ϵ^- on $\Omega \setminus \mathcal{T}_\epsilon^-$:

$$(7.3) \quad v_\epsilon^-(x) = \begin{cases} 1 + \epsilon \eta_\infty g(x) + \epsilon^2 \xi_\infty^{(2)} (g(x))^2 - c_2 \epsilon^3 |\log \epsilon|^2 & \text{if } d_\epsilon^-(x) < -2\delta\epsilon |\log \epsilon|, \\ -1 + \epsilon \eta_\infty g(x) - \epsilon^2 \xi_\infty^{(2)} (g(x))^2 - c_2 \epsilon^3 |\log \epsilon|^2 & \text{if } d_\epsilon^-(x) > 2\delta\epsilon |\log \epsilon|, \end{cases}$$

where we also exploit $\eta_{-\infty} = \eta_\infty$ and $\xi_{-\infty}^{(2)} = -\xi_\infty^{(2)}$.

The two definitions agree on the common set $\mathcal{T} \setminus \mathcal{T}_\epsilon^-$.

7.1. Subsolution on \mathcal{T}_ϵ^- . The most delicate part of this paper is to verify that v_ϵ^- is actually a subsolution on \mathcal{T}_ϵ^- with an appropriate choice of the constants c_1, c_2 . Note that

$$(7.4) \quad d, d_\epsilon^- = \mathcal{O}(\epsilon |\log \epsilon|).$$

We need the following lemma; the proof can be found in [22, Theorem 3.2].

Lemma 7.1. *Given any function $w \in C^2(\Sigma)$, we can compute the Laplace-Beltrami operator $\Delta_\Sigma w$ as follows:*

$$\Delta_\Sigma w(s(x)) = \Delta(w \circ s)(x), \quad \forall x \in \Sigma,$$

where we endow Σ with the Riemannian metric induced by \mathbf{R}^n .

For $x \in \mathcal{T}_\epsilon^-$ we apply Lemma 7.1 to the eigenfunction h . By Taylor expansion of $\Delta \bar{h}(x)$ about $s(x)$ we get

$$(7.5) \quad \Delta \bar{h}(x) = \Delta_\Sigma h(s(x)) + d\nabla(\Delta \bar{h}(\xi)) \cdot \nabla d = \Delta_\Sigma h(s(x)) + \mathcal{O}(\epsilon |\log \epsilon|) \quad \forall x \in \mathcal{T}_\epsilon^-,$$

where the constant in $\mathcal{O}(\epsilon |\log \epsilon|)$ involves three derivatives of h and Σ .

The following relation holds in \mathcal{T}_ϵ^- [20, §14.6]:

$$\Delta d(x) = -\bar{\kappa}^m(x) - d(x) \bar{\kappa}^s(x) + \mathcal{O}(d^2(x)).$$

We can use (7.5) to express the Laplacian of the modified distance as

$$(7.6) \quad \begin{aligned} \Delta d_\epsilon^- &= \Delta d - c_1 \Delta_\Sigma h(s) \epsilon^2 |\log \epsilon|^2 + c_1 \mathcal{O}(d \epsilon^2 |\log \epsilon|^2) \\ &= -\bar{\kappa}^m - \epsilon y \bar{\kappa}^s - c_1 (\bar{h} \bar{\kappa}^s + \Delta_\Sigma h(s)) \epsilon^2 |\log \epsilon|^2 + \mathcal{O}(\epsilon^2 |\log \epsilon|^2). \end{aligned}$$

The derivation, from here on, strictly follows that of [3, §6.3]. Direct computations yield

$$\nabla v_\epsilon^- = \epsilon^{-1} \gamma_\epsilon' \nabla d_\epsilon^- + g \eta_\epsilon' \nabla d_\epsilon^- + \epsilon \mathbf{b} \cdot \xi_\epsilon' \nabla d_\epsilon^- + \epsilon \eta_\epsilon \nabla g + \epsilon^2 \xi_\epsilon \cdot \nabla \mathbf{b},$$

where $\nabla \mathbf{b}$ acts componentwise. Using (7.1) and (2.1) to bound $\nabla \mathbf{b}$ and $\Delta \mathbf{b}$, and noting that $c_1^2 \rho(\epsilon^\alpha) = \mathcal{O}(\epsilon^\alpha)$, we get

$$\Delta v_\epsilon^- = \epsilon^{-2} \gamma_\epsilon'' + \epsilon^{-1} g \eta_\epsilon'' + \epsilon^{-1} \gamma_\epsilon' \Delta d_\epsilon^- + g \eta_\epsilon' \Delta d_\epsilon^- + \mathbf{b} \cdot \xi_\epsilon'' + 2 \eta_\epsilon' \nabla g \cdot \nabla d_\epsilon^- + \mathcal{O}(\epsilon).$$

Hence, using (7.6), we have

$$\begin{aligned} -\epsilon^2 \Delta v_\epsilon^- &= -\gamma_\epsilon'' - \epsilon g \eta_\epsilon'' - \epsilon^2 \mathbf{b} \cdot \xi_\epsilon'' - (\epsilon \gamma_\epsilon' + \epsilon^2 g \eta_\epsilon') \Delta d_\epsilon^- - 2 \epsilon^2 \eta_\epsilon' \nabla g \cdot \nabla d_\epsilon^- + \mathcal{O}(\epsilon^3) \\ &= -\gamma_\epsilon'' - \epsilon g \eta_\epsilon'' + \epsilon \gamma_\epsilon' \bar{\kappa}^m - \epsilon^2 \mathbf{b} \cdot \xi_\epsilon'' - 2 \epsilon^2 \eta_\epsilon' \nabla g \cdot \nabla d_\epsilon^- + \epsilon^2 g \bar{\kappa}^m \eta_\epsilon' + \epsilon^2 y \gamma_\epsilon' \bar{\kappa}^s \\ &\quad + c_1 \epsilon^3 |\log \epsilon|^2 \gamma_\epsilon' (\bar{h} \bar{\kappa}^s + \Delta_\Sigma h(s)) + \mathcal{O}(\epsilon^3 |\log \epsilon|^2). \end{aligned}$$

Now observe that, by (4.1) and noting that $\nabla d = \nabla d_\epsilon^- + c_1 \mathcal{O}(\epsilon^2 |\log \epsilon|^2)$, we obtain

$$\begin{aligned} g(x) &= g(s(x) + d \nabla d) \\ &= \bar{\kappa}^m + d \nabla d \cdot \overline{\nabla g} + \mathcal{O}(d^2) \\ &= \bar{\kappa}^m + \epsilon y \nabla d_\epsilon^- \cdot \overline{\nabla g} + c_1 \bar{h} \epsilon^2 |\log \epsilon|^2 \nabla d_\epsilon^- \cdot \overline{\nabla g} + \mathcal{O}(\epsilon^2 |\log \epsilon|^2), \end{aligned}$$

where $\overline{\nabla g}(x) = (\nabla g)(s(x))$. Since $y = \mathcal{O}(|\log \epsilon|^2)$, it thus follows that

$$(7.7) \quad \epsilon^2 g \bar{\kappa}^m \eta_\epsilon' = \epsilon^2 g^2 \eta_\epsilon' + \mathcal{O}(\epsilon^3 |\log \epsilon|),$$

and

$$(7.8) \quad \epsilon \gamma_\epsilon' \bar{\kappa}^m = \epsilon \gamma_\epsilon' g - \epsilon^2 y \gamma_\epsilon' \nabla d_\epsilon^- \cdot \overline{\nabla g} - c_1 \bar{h} \epsilon^3 |\log \epsilon|^2 \gamma_\epsilon' \nabla d_\epsilon^- \cdot \overline{\nabla g} + \mathcal{O}(\epsilon^3 |\log \epsilon|^2).$$

Inserting (7.7) and (7.8) in the previous expression of $-\epsilon^2 \Delta v_\epsilon^-$, we get

$$\begin{aligned} -\epsilon^2 \Delta v_\epsilon^- &= -\gamma_\epsilon'' - \epsilon g \eta_\epsilon'' + \epsilon \gamma_\epsilon' g - \epsilon^2 y \gamma_\epsilon' \nabla d_\epsilon^- \cdot \overline{\nabla g} - c_1 \bar{h} \epsilon^3 |\log \epsilon|^2 \gamma_\epsilon' \nabla d_\epsilon^- \cdot \overline{\nabla g} \\ &\quad - \epsilon^2 \mathbf{b} \cdot \xi_\epsilon'' - 2 \epsilon^2 \eta_\epsilon' \nabla g \cdot \nabla d_\epsilon^- + \epsilon^2 g^2 \eta_\epsilon' + \epsilon^2 y \gamma_\epsilon' \bar{\kappa}^s \\ &\quad + c_1 \epsilon^3 |\log \epsilon|^2 \gamma_\epsilon' (\bar{h} \bar{\kappa}^s + \Delta_\Sigma h(s)) + \mathcal{O}(\epsilon^3 |\log \epsilon|^2). \end{aligned}$$

Using the formula

$$\begin{aligned} \psi(v_\epsilon^-) &= \psi(\gamma_\epsilon) + \epsilon g \eta_\epsilon \psi'(\gamma_\epsilon) + \epsilon^2 \mathbf{b} \cdot \xi_\epsilon \psi'(\gamma_\epsilon) \\ &\quad + \frac{1}{2} \epsilon^2 g^2 \eta_\epsilon^2 \psi''(\gamma_\epsilon) - c_2 \epsilon^3 |\log \epsilon|^2 \psi'(\gamma_\epsilon) + \mathcal{O}(\epsilon^3), \end{aligned}$$

we finally get that

$$-\epsilon^2 \Delta v_\epsilon^- + \psi(v_\epsilon^-) - \epsilon \frac{c_0}{2} g = \text{I}_\epsilon + \epsilon \text{II}_\epsilon + \epsilon^2 \text{III}_\epsilon + \epsilon^3 |\log \epsilon|^2 \text{IV}_\epsilon + \mathcal{O}(\epsilon^3 |\log \epsilon|^2) \text{ in } \mathcal{T}_\epsilon^-,$$

where, recalling the expression of \mathbf{b} and ξ_ϵ ,

$$\begin{aligned} \text{I}_\epsilon &= -(\gamma_\epsilon'' - \psi(\gamma_\epsilon)), \\ \text{II}_\epsilon &= -g \left[\eta_\epsilon'' - \eta_\epsilon \psi'(\gamma_\epsilon) + \frac{c_0}{2} - \gamma_\epsilon' \right], \\ \text{III}_\epsilon &= -\mathbf{b} \cdot \xi_\epsilon'' + \mathbf{b} \cdot \xi_\epsilon \psi'(\gamma_\epsilon) - 2\eta_\epsilon' \nabla g \cdot \nabla d + g^2 \eta_\epsilon' + y \gamma_\epsilon' \bar{\kappa}^s + \frac{1}{2} g^2 \eta_\epsilon^2 \psi''(\gamma_\epsilon) - y \gamma_\epsilon' \nabla d \cdot \overline{\nabla g} \\ &= -(\bar{\kappa}^s - \nabla d \cdot \nabla g)(\xi_\epsilon^{(3)''} - \psi'(\gamma_\epsilon) \xi_\epsilon^{(3)} - y \gamma_\epsilon') + y \gamma_\epsilon' \nabla d \cdot (\nabla g - \overline{\nabla g}) \\ &\quad - (g^2 - 2\nabla d \cdot \nabla g)(\xi_\epsilon^{(1)''} - \psi'(\gamma_\epsilon) \xi_\epsilon^{(1)} - \eta_\epsilon') - g^2(\xi_\epsilon^{(2)''} - \psi'(\gamma_\epsilon) \xi_\epsilon^{(2)} - \frac{1}{2} \eta_\epsilon^2 \psi''(\gamma_\epsilon)), \\ \text{IV}_\epsilon &= c_1 \gamma_\epsilon' (\bar{h} \bar{\kappa}^s + \Delta_\Sigma h(s)) - c_2 \psi'(\gamma_\epsilon) - c_1 \bar{h} \gamma_\epsilon' \nabla d \cdot \overline{\nabla g} = -c_1 \lambda_1 \bar{h} \gamma_\epsilon' - c_2 \psi'(\gamma_\epsilon), \end{aligned}$$

where we also use the equality $\nabla d_\epsilon^- = \nabla d + c_1 \mathcal{O}(\epsilon^2 |\log \epsilon|^2) = \nabla d + \mathcal{O}(\epsilon)$ and relation (4.3) for the eigenfunction h , with $\nabla d = -n$.

Observe now that [3] in \mathcal{T}_ϵ^-

$$(7.9) \quad \begin{aligned} \gamma_\epsilon'' - \psi(\gamma_\epsilon) &= o(\epsilon^{2\delta-1}), \\ \eta_\epsilon'' - \psi'(\gamma_\epsilon) \eta_\epsilon + \frac{c_0}{2} - \gamma_\epsilon' &= o(\epsilon^{2\delta-1}), \\ \xi_\epsilon^{(3)''} - \psi'(\gamma_\epsilon) \xi_\epsilon^{(3)} - y \gamma_\epsilon' &= \xi_\epsilon^{(1)''} - \psi'(\gamma_\epsilon) \xi_\epsilon^{(1)} - \eta_\epsilon' \\ &= \xi_\epsilon^{(2)''} - \psi'(\gamma_\epsilon) \xi_\epsilon^{(2)} - \frac{1}{2} \eta_\epsilon^2 \psi''(\gamma_\epsilon) = o(\epsilon^{2\delta-1}), \end{aligned}$$

for any $y \neq \pm y_\epsilon$. Noting that $y \gamma_\epsilon' \nabla d \cdot (\nabla g - \overline{\nabla g}) = \mathcal{O}(\epsilon |\log \epsilon|^2)$, we conclude from the previous estimates that

$$\text{I}_\epsilon + \epsilon \text{II}_\epsilon + \epsilon^2 \text{III}_\epsilon = o(\epsilon^{2\delta-1}) + \mathcal{O}(\epsilon^3 |\log \epsilon|^2) \quad \text{in } \mathcal{T}_\epsilon^-.$$

Since $\lim_{y \rightarrow \pm\infty} \psi'(\gamma_\epsilon(y)) = \psi'(\pm 1) = 4$, there exists a positive constant c_3 such that $c_3 \gamma_\epsilon' + \psi'(\gamma_\epsilon) \geq 2$ on \mathbf{R} . By choosing c_2 large enough and

$$c_1 = \frac{c_3 c_2}{\lambda_1 h_{\min}} > 0$$

we have

$$(7.10) \quad \begin{aligned} \mathcal{L}(v_\epsilon^-) &= -\epsilon^2 \Delta v_\epsilon^- + \psi(v_\epsilon^-) - \epsilon \frac{c_0}{2} g = -\epsilon^3 |\log \epsilon|^2 (c_1 \lambda_1 \bar{h} \gamma_\epsilon' + c_2 \psi'(\gamma_\epsilon) + \mathcal{O}(1)) \\ &\leq -\epsilon^3 |\log \epsilon|^2 (c_3 c_2 \gamma_\epsilon' + c_2 \psi'(\gamma_\epsilon) + \mathcal{O}(1)) \leq -C \epsilon^3 |\log \epsilon|^2 < 0 \end{aligned}$$

for some constant $C > 0$, since the constant in $\mathcal{O}(1)$ does not depend on the constants c_1, c_2 . Note that the constant c_1 , which appears in the final error estimate, depends in particular on the nondegeneracy of the minimizer A , given by the size of the positive eigenvalue λ_1 .

7.2. Subsolution on $\Omega \setminus \mathcal{T}_\epsilon^-$. Suppose, for definiteness, that $d_\epsilon^- < -2\delta\epsilon |\log \epsilon|$. From (2.1) it follows immediately that $-\epsilon^2 \Delta v_\epsilon^- = \mathcal{O}(\epsilon^3)$ and, using (6.2), we get

$$\begin{aligned} \psi(v_\epsilon^-) &= \psi(1) + \epsilon \eta_\infty g \psi'(1) + \epsilon^2 \xi_\infty^{(2)} g^2 \psi'(1) - c_2 \epsilon^3 |\log \epsilon|^2 \psi'(1) \\ &\quad + \frac{1}{2} \epsilon^2 \eta_\infty^2 g^2 \psi''(1) + \mathcal{O}(\epsilon^3) \\ &= \epsilon \frac{c_0}{2} g - c_2 \epsilon^3 |\log \epsilon|^2 \psi'(1) + \mathcal{O}(\epsilon^3). \end{aligned}$$

Finally,

$$(7.11) \quad \begin{aligned} \mathcal{L}(v_\epsilon^-) &= -\epsilon^2 \Delta v_\epsilon^- + \psi(v_\epsilon^-) - \epsilon \frac{c_0}{2} g = -c_2 \epsilon^3 |\log \epsilon|^2 \psi'(1) + \mathcal{O}(\epsilon^3) \\ &\leq -C \epsilon^3 |\log \epsilon|^2 < 0, \end{aligned}$$

provided we choose c_2 sufficiently large.

7.3. Comparison with boundary datum. Unfortunately, the constructed subsolution v_ϵ^- has boundary value given by the second line of (7.3), which is not comparable with the Dirichlet boundary value -1 , for a general g . Unless we require $g = 0$ on $\partial\Omega$, we are then forced to suitably modify the subsolution v_ϵ^- in a tubular neighborhood of $\partial\Omega$, and we then need appropriate regularity ($W^{3,\infty}$) of $\partial\Omega$.

We shall denote by \hat{d} the (positive) distance function of any point $x \in \Omega$ to the boundary, and by $\hat{y} = \hat{d}\epsilon^{-1}$ the corresponding stretched variable. Let $\hat{\mathcal{T}}_\epsilon = \{x \in \Omega : \hat{d}(x) \leq 2\hat{\delta}\epsilon |\log \epsilon|\}$ for some positive constant $\hat{\delta}$, which is disjoint from \mathcal{T}_ϵ^- for ϵ small. The shape of v_ϵ^- in $\hat{\mathcal{T}}_\epsilon$ is again suggested by formal asymptotics (see §6.3). Because of the matching condition with v_ϵ^- in the interior of Ω we need to use the first three terms of the formal expansion. More precisely, we define, similarly to (7.2),

$$v_\epsilon^-(x) = -1 + \epsilon g(x) \hat{\eta}_\epsilon(\hat{y}) + \epsilon^2 \hat{\mathbf{b}} \cdot \hat{\xi}_\epsilon(\hat{y}) - c_2 \epsilon^3 |\log \epsilon|^2, \quad \forall x \in \hat{\mathcal{T}}_\epsilon,$$

where $\hat{\mathbf{b}} = (\hat{\kappa}^m g - 2\nabla g \cdot \nabla \hat{d}, g^2)$, $\hat{\xi}_\epsilon = (\hat{\xi}_\epsilon^{(1)}, \hat{\xi}_\epsilon^{(2)})$, and $\hat{\kappa}^m$ refers to the curvature of $\partial\Omega$. The shape functions $\hat{\eta}_\epsilon, \hat{\xi}_\epsilon^{(1)}, \hat{\xi}_\epsilon^{(2)}$ can be explicitly constructed and are given by

$$(7.12) \quad \begin{aligned} \hat{\eta}_\epsilon(\hat{y}) &= \frac{c_0}{2\psi'(1)} (1 - \exp_\epsilon(-\sigma\hat{y})), \\ \hat{\xi}_\epsilon^{(1)}(\hat{y}) &= -\frac{c_0}{4\psi'(1)} \hat{y} \exp_\epsilon(-\sigma\hat{y}), \\ \hat{\xi}_\epsilon^{(2)}(\hat{y}) &= \frac{c_0^2 \psi''(1)}{8[\psi'(1)]^3} \left[1 - \left(\frac{2}{3} + \sigma\hat{y} \right) \exp_\epsilon(-\sigma\hat{y}) - \frac{1}{3} (\exp_\epsilon(-\sigma\hat{y}))^2 \right], \end{aligned}$$

where $\sigma = \sqrt{\psi'(1)}$ and \exp_ϵ denotes the exponential function modified for $\hat{y} > \hat{\delta} |\log \epsilon|$ by a cubic polynomial, as done for the “internal layer” shape functions, so as to vanish for $\hat{y} \geq 2\hat{\delta} |\log \epsilon|$.

It is clear from this definition that $v_\epsilon^-(x) < -1$ for any $x \in \partial\Omega$ and that there is consistency with definition (7.3). It only remains to prove that $\mathcal{L}v_\epsilon^- \leq -C\epsilon^3 |\log \epsilon|^2 < 0$ in $\hat{\mathcal{T}}_\epsilon$. Indeed, if we expand $\mathcal{L}v_\epsilon^-$ as we did in \mathcal{T}_ϵ^- , we find that the terms of order $\mathcal{O}(1)$, $\mathcal{O}(\epsilon)$, $\mathcal{O}(\epsilon^2)$ essentially vanish, apart for contributions of order $\mathcal{O}(\epsilon^3)$ owing to the truncation of the exponential for $y > \hat{\delta} |\log \epsilon|$; we remain with the $\mathcal{O}(\epsilon^3 |\log \epsilon|^2)$ term, which has the form $-c_2 \psi'(1) \epsilon^3 |\log \epsilon|^2 < 0$.

7.4. The approximating solution. The previous arguments can be repeated to construct a supersolution v_ϵ^+ based on the modified distance $d_\epsilon^+(x) = d(x) + c_1 \bar{h}(x) \epsilon^2 |\log \epsilon|^2$.

We can now apply the Comparison Lemma 5.1 and deduce that there exists a relative minimum u_ϵ of \mathcal{F}_ϵ satisfying the inequality

$$v_\epsilon^- \leq u_\epsilon \leq v_\epsilon^+ \quad \text{in } \Omega.$$

8. INTERFACE ERROR ESTIMATE

Let us denote by dist_H the Hausdorff distance between subsets of \mathbf{R}^n , namely

$$\text{dist}_H(E, F) = \max\left\{\sup_{x \in F} \text{dist}(x, E), \sup_{x \in E} \text{dist}(x, F)\right\}.$$

Note that dist_H satisfies the following elementary properties:

If $E \subseteq F$ are two different subsets of \mathbf{R}^n , then

$$(8.1) \quad \text{dist}_H(E, F) = \sup_{x \in F} \text{dist}(x, E) = \sup_{x \in F \setminus E} \text{dist}(x, E) = \sup_{x \in F \setminus E} \text{dist}(x, \partial E).$$

If $A^- \subseteq E, F \subseteq A^+$, then

$$(8.2) \quad \text{dist}_H(\partial E, \partial F) \leq \max\left\{\text{dist}_H(A^-, A^+), \text{dist}_H(\mathbf{R}^n \setminus A^-, \mathbf{R}^n \setminus A^+)\right\}.$$

In fact, if $x \in \partial F \setminus E$, then

$$\text{dist}(x, \partial E) = \text{dist}(x, E) \leq \text{dist}(x, A^-) \leq \text{dist}_H(A^-, A^+).$$

If $x \in \partial F \cap E$, then

$$\text{dist}(x, \partial E) = \text{dist}(x, \mathbf{R}^n \setminus E) \leq \text{dist}(x, \mathbf{R}^n \setminus A^+) \leq \text{dist}_H(\mathbf{R}^n \setminus A^-, \mathbf{R}^n \setminus A^+).$$

This reasoning can then be repeated, with the roles of E and F interchanged.

From now on we shall denote by C a positive constant, different from time to time and independent of ϵ .

Theorem 8.1. *Assume the regularity assumptions (2.1) and either $\partial\Omega \in W^{3,\infty}$ (i.e., $\hat{d} \in W^{3,\infty}(\hat{\mathcal{T}}_\epsilon)$) or g compactly supported in Ω . Given a nondegenerate relative minimizer $A \subset\subset \Omega$ of $\tilde{\mathcal{F}}$, there exist a solution u_ϵ of (3.2), a constant C independent of ϵ but possibly depending on Ω, g, A (in particular on λ_1), and a constant $\epsilon_0 > 0$ such that*

$$\text{dist}_H(\Sigma, \Sigma_\epsilon) \leq C\epsilon^2 |\log \epsilon|^2, \quad \forall \epsilon < \epsilon_0,$$

where $\Sigma = \partial A$ and $\Sigma_\epsilon = \{u_\epsilon = 0\}$.

Proof. Existence of u_ϵ is established in §7.4. Set $A^- = \{\gamma(\frac{d_\epsilon^-}{\epsilon}) - C_1\epsilon \geq 0\}$, $A^+ = \{\gamma(\frac{d_\epsilon^+}{\epsilon}) + C_1\epsilon \geq 0\}$, $A_\epsilon = \{u_\epsilon \geq 0\}$. Choosing C_1 sufficiently large (independent of ϵ), we clearly have $\gamma(\frac{d_\epsilon^-}{\epsilon}) - C_1\epsilon \leq v_\epsilon^-$ and $\gamma(\frac{d_\epsilon^+}{\epsilon}) + C_1\epsilon \geq v_\epsilon^+$. Hence, $A^- \subseteq A, A_\epsilon \subseteq A^+$, and we can use (8.2). Now, for ϵ sufficiently small, $A^+ \setminus A^- \subseteq \mathcal{T}$, and we can conclude that $\text{dist}_H(A^-, A^+), \text{dist}_H(\mathbf{R}^n \setminus A^-, \mathbf{R}^n \setminus A^+) \leq C\epsilon^2 |\log \epsilon|^2$. To see this, we argue as follows. Fix $x \in A^+ \setminus A^-$. We have $\gamma(\frac{d_\epsilon^+}{\epsilon}) + C_1\epsilon \geq 0$ and $\gamma(\frac{d_\epsilon^-}{\epsilon}) - C_1\epsilon < 0$, which imply

$$d_\epsilon^+ \geq -\epsilon\gamma^{-1}(C_1\epsilon) \geq -2C_1\epsilon^2, \quad d_\epsilon^- < 2C_1\epsilon^2.$$

Define $y = x + C\epsilon^2 |\log \epsilon|^2 \nabla d$ with $C > 3C_1 + 2c_1 h_{\max}$. From the properties of the distance function and the definition of d_ϵ^- and d_ϵ^+ we clearly have $s(x) = s(y)$, and we readily get $d_\epsilon^-(y) = d_\epsilon^+(y) - 2c_1 \bar{h}(x)\epsilon^2 |\log \epsilon|^2 > d_\epsilon^+(x) + 3C_1\epsilon^2 > 2C_1\epsilon^2$. This implies $\frac{d_\epsilon^-(y)}{\epsilon} > 2C_1\epsilon$, which gives $y \in A^-$, since $\gamma(2C_1\epsilon) > C_1\epsilon$ for ϵ sufficiently small. Since $\text{dist}(x, y) = C\epsilon^2 |\log \epsilon|^2$, we get the result, using (8.1). \square

9. OPTIMALITY

Optimality of the quadratic rate of convergence expressed in Theorem 8.1 can be formally shown by examining the very special case of a circular minimizer, and using formal asymptotics.

Take $n = 2$, and select a radially symmetric forcing term g of the special form

$$g(r) = \alpha r^2 + \beta r + 1 - \alpha - \beta,$$

where $r = |x|$ is the distance from the origin. Since $g(1) = 1$, the circle $A = \{r < 1\}$ is a stationary point for the functional $\tilde{\mathcal{F}}$ in (3.1). It is not difficult to verify that this circle is the unique absolute minimizer of $\tilde{\mathcal{F}}$ for all (α, β) in some neighborhood of $(0, -4)$.

If we express (3.2) for radial symmetry, we obtain for the solution $u_\epsilon = u_\epsilon(r)$ the equation

$$(9.1) \quad \text{I} + \text{II} + \text{III} + \text{IV} := -\epsilon^2 r u_\epsilon'' - \epsilon^2 u_\epsilon' + r \psi(u_\epsilon) - \frac{c_0}{2} \epsilon r g(r) = 0.$$

We shall denote by ϕ_ϵ a root of $u_\epsilon(r)$, i.e., $u_\epsilon(\phi_\epsilon) = 0$, which corresponds to the radius of the approximating circle Σ_ϵ . Set $\phi_\epsilon = 1 + \sigma_\epsilon$.

If we suppose by contradiction that the error between Σ and Σ_ϵ is more than quadratic in ϵ , we should have

$$\sigma_\epsilon = o(\epsilon^2).$$

Finally, we introduce the stretched variable $y = \epsilon^{-1}(\phi_\epsilon - r)$, and the stretched solution within the transition interval,

$$U_\epsilon(y) = u_\epsilon(\phi_\epsilon - \epsilon y).$$

We now intend to develop an accurate formal asymptotic expansion for U_ϵ and consider all the terms of order $\geq \mathcal{O}(\epsilon^3)$. Expand U_ϵ as follows:

$$(9.2) \quad U_\epsilon = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \epsilon^3 U_3 + o(\epsilon^3).$$

Since $U_\epsilon(0) = 0$, we get the conditions

$$(9.3) \quad U_i(0) = 0, \quad i = 0, \dots, 3.$$

We clearly have

$$(9.4) \quad r = 1 - \epsilon y + \sigma_\epsilon,$$

$$(9.5) \quad u_\epsilon' = -\frac{1}{\epsilon} U_\epsilon',$$

$$(9.6) \quad u_\epsilon'' = \frac{1}{\epsilon^2} U_\epsilon'',$$

$$(9.7) \quad g(r) = 1 - \epsilon(2\alpha + \beta)y + \epsilon^2 \alpha y^2 + o(\epsilon^2),$$

$$\begin{aligned} \psi(U_\epsilon) &= \psi(U_0) + (U_\epsilon - U_0)\psi'(U_0) + \frac{1}{2}(U_\epsilon - U_0)^2\psi''(U_0) \\ &\quad + \frac{1}{6}(U_\epsilon - U_0)^3\psi'''(U_0) + o(\epsilon^3) \end{aligned}$$

$$(9.8) \quad \begin{aligned} &= \psi(U_0) + \epsilon U_1 \psi'(U_0) + \epsilon^2 \left(U_2 \psi'(U_0) + \frac{1}{2} U_1^2 \psi''(U_0) \right) \\ &\quad + \epsilon^3 \left(U_3 \psi'(U_0) + U_1 U_2 \psi''(U_0) + \frac{1}{6} U_1^3 \psi'''(U_0) \right) + o(\epsilon^3). \end{aligned}$$

We can now use these relations to obtain the expansions for each term in (9.1). Using (9.2), (9.4), and (9.6), we get

$$(9.9) \quad \begin{aligned} \text{I} &= -\epsilon^2(1 - \epsilon y + \sigma_\epsilon) \frac{1}{\epsilon^2} (U_0'' + \epsilon U_1'' + \epsilon^2 U_2'' + \epsilon^3 U_3'' + o(\epsilon^3)) \\ &= -U_0'' + \epsilon(yU_0'' - U_1'') + \epsilon^2(yU_1'' - U_2'') - \sigma_\epsilon U_0'' + \epsilon^3(yU_2'' - U_3'') + o(\epsilon^3). \end{aligned}$$

Using (9.2) and (9.5), we get

$$(9.10) \quad \text{II} = -\epsilon^2 \left(-\frac{1}{\epsilon} (U_0' + \epsilon U_1' + \epsilon^2 U_2' + o(\epsilon^2)) \right) = \epsilon U_0' + \epsilon^2 U_1' + \epsilon^3 U_2' + o(\epsilon^3).$$

Using (9.2), (9.4), and (9.8), we get

$$(9.11) \quad \begin{aligned} \text{III} &= (1 - \epsilon y + \sigma_\epsilon) \psi(U_\epsilon) \\ &= \psi(U_0) + \epsilon \left(U_1 \psi'(U_0) - y \psi(U_0) \right) + \epsilon^2 \left(U_2 \psi'(U_0) + \frac{1}{2} U_1^2 \psi''(U_0) - y U_1 \psi'(U_0) \right) \\ &\quad + \epsilon^3 \left(U_3 \psi'(U_0) + U_1 U_2 \psi''(U_0) + \frac{1}{6} U_1^3 \psi'''(U_0) - y U_2 \psi'(U_0) - \frac{1}{2} y U_1^2 \psi''(U_0) \right) \\ &\quad + o(\epsilon^3) + \sigma_\epsilon \psi(U_0). \end{aligned}$$

Using (9.4) and (9.7), we get

$$(9.12) \quad \begin{aligned} \text{IV} &= -\frac{c_0}{2} \epsilon (1 - \epsilon y + \sigma_\epsilon) \left(1 - \epsilon(2\alpha + \beta)y + \epsilon^2 \alpha y^2 + o(\epsilon^2) \right) \\ &= \frac{c_0}{2} \left(-\epsilon + \epsilon^2 y(2\alpha + \beta + 1) - \epsilon^3 y^2(3\alpha + \beta) \right) + o(\epsilon^3). \end{aligned}$$

We now collect all terms of equal order in (9.9), (9.10), (9.11), (9.12) and equate them to zero.

9.1. Order $\mathcal{O}(1)$. Not surprisingly, the $\mathcal{O}(1)$ terms from (9.9), (9.11), together with condition (9.3), force the choice $U_0 = \gamma$. This also fortunately cancels out the two terms with σ_ϵ in the same two expressions for I and III.

9.2. Order $\mathcal{O}(\epsilon)$. After cancelling two terms, we get the equation for U_1 , recalling (6.1),

$$\mathcal{A}U_1 = \gamma' - \frac{c_0}{2},$$

whence, as expected, $U_1 = \eta$.

9.3. Order $\mathcal{O}(\epsilon^2)$. We get

$$\begin{aligned} \mathcal{A}U_2 &= y\mathcal{A}U_1 + \eta' + \frac{1}{2} \eta^2 \psi''(\gamma) + \frac{c_0}{2} y(2\alpha + \beta + 1) \\ &= \eta' + \frac{1}{2} \eta^2 \psi''(\gamma) + y\gamma' + \frac{c_0}{2} y(2\alpha + \beta). \end{aligned}$$

This leads to $U_2 = \xi^{(1)} + \xi^{(2)} + \xi^{(3)} - (2\alpha + \beta)\xi^{(4)}$.

9.4. Order $\mathcal{O}(\epsilon^3)$. We are now only interested in terms involving α and β . All terms not depending on α or β will always be denoted by the generic letter f . We obtain, recalling that $\mathcal{A}U_2 = \frac{c_0}{2}y(2\alpha + \beta) + f$,

$$\begin{aligned} \mathcal{A}U_3 &= \frac{c_0}{2}y^2(2\alpha + \beta) - (2\alpha + \beta)(\xi^{(4)})' - \eta(2\alpha + \beta)\xi^{(4)}\psi''(\gamma) - \frac{c_0}{2}y^2(3\alpha + \beta) + f \\ &= -\frac{c_0}{2}\alpha y^2 - (2\alpha + \beta)((\xi^{(4)})' + \eta\xi^{(4)}\psi''(\gamma)) + f =: f_{\alpha,\beta}. \end{aligned}$$

The solvability condition for this self-adjoint equation (see, e.g., [3]) implies

$$(9.13) \quad \int_{\mathbf{R}} f_{\alpha,\beta}\gamma' = 0,$$

and this has to be true for all (α, β) in a neighborhood of $(0, -4)$; hence, by differentiating (9.13) with respect to α , keeping $2\alpha + \beta$ fixed, we get

$$-\frac{c_0}{2} \int_{\mathbf{R}} y^2\gamma' = 0,$$

but this is not possible, since $\gamma' > 0$, and we get a contradiction. This proves that the error between Σ and Σ_ϵ cannot in general be better than $\mathcal{O}(\epsilon^2)$, and that our main result is quasi-optimal.

10. THE DISCRETE PROBLEM

Using conforming piecewise linear finite elements, we shall now introduce a discretized version of problem (3.2)–(3.3) equivalent to a system of nonlinear equations, which can be numerically solved, employing appropriate iterative schemes.

We shall prove, under appropriate restrictions on the mesh size ($h^2 = \mathcal{O}(\epsilon^5)$) a discrete counterpart of Theorem 8.1, e.g., that the boundary Σ of the nondegenerate minimizer $A \subset \subset \Omega$ can be approximated with an $\mathcal{O}(\epsilon^2 |\log \epsilon|^2)$ error by a solution of the discretized problem.

10.1. Notations and assumptions. For the sake of simplicity we shall assume Ω to be a convex polyhedral domain in \mathbf{R}^n . Since this implies that $\partial\Omega$ is not regular, we shall require g to be compactly supported in Ω .

For any $h > 0$ (meshsize) let $\{\mathcal{S}_h\}_h$ be a family of finite element partitions of Ω into simplices having diameter bounded by h and satisfying the minimum-angle regularity assumption [10]. It is not restrictive to assume $\bar{\Omega} = \bigcup_{T \in \mathcal{S}_h} T$. Denote by $\mathcal{N}_h = \{N_{ij}\}_{i=1}^J$ the set of all vertices of the mesh and suppose that $\mathcal{N}_h^0 = \{N_{ij}\}_{i=1}^I$, $I < J$, are the internal vertices.

For the main result of this section we also require the following regularity assumption on \mathcal{S}_h :

- There exists a constant C independent of h such that for any tetrahedron
- (A) $T \in \mathcal{S}_h$ the projection of any vertex lies inside the opposite face and has distance bigger than Ch from the boundary of the opposite face.

In dimension $n = 2$, assumption (A) can be replaced by the less restrictive requirement:

- (A*) There exists a constant C independent of h such that for any pair of adjacent triangles $T_1, T_2 \in \mathcal{S}_h$ the sum of the opposite angles does not exceed $\pi - C$.

Assumption (A) (respectively (A_{*}) in dimension $n = 2$) is a stronger version of the *acuteness* property of meshes (respectively *quasi-acuteness*), and implies a discrete version of the comparison principle for the discrete operator without mass lumping (Lemma 10.1).

We shall denote by V_h the space of globally continuous piecewise linear functions over \mathcal{S}_h , and denote $V_h^0 := V_h \cap H_0^1(\Omega) = \{v \in V_h : v|_{\partial\Omega} = 0\}$. Let $\phi_j \in V_h$ denote the usual hat function associated with the node N_j , $j = 1, \dots, J$. The functions ϕ_j , $j = 1, \dots, J$ form the standard canonical basis for the finite element space V_h . Note that the subset $\{\phi_i\}_{i=1}^J$ is a basis for $V_h^0 \subset V_h$. Finally, $I_h : C^0(\bar{\Omega}) \rightarrow V_h$ will denote the usual Lagrange interpolation operator, and Π_h the elliptic projection of any $v \in H^1(\Omega)$ onto V_h defined by

$$(10.1) \quad (\nabla v, \nabla \phi) = (\nabla(\Pi_h v), \nabla \phi), \quad \forall \phi \in V_h^0, \quad \Pi_h v = I_h v \text{ on } \partial\Omega.$$

Denote by $M = (m_{ij})$ and by $A = (a_{ij})$ the mass and stiffness matrices, defined by

$$m_{ij} = \int_{\Omega} \phi_i \phi_j \, dx, \quad a_{ij} = \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j \, dx, \quad i = 1, \dots, I, \quad j = 1, \dots, J.$$

It follows by direct computation that for meshes satisfying assumption (A), or (A_{*}) if $n = 2$, we have

$$(10.2) \quad -a_{ij} \geq Ch^{-2}m_{ij} \geq 0, \quad i = 1, \dots, I, \quad j = 1, \dots, J, \quad i \neq j,$$

where C denotes here and below some positive constant independent of h and ϵ , possibly different from line to line.

The discrete version of problem (3.2)–(3.3) can be written in variational form as

$$(10.3) \quad \text{find } v \in K_h \text{ such that} \\ (\mathcal{L}_h(v), \phi) := \epsilon^2(\nabla v, \nabla \phi) + (I_h \psi(v), \phi) - \epsilon \frac{C_0}{2}(g, \phi)_h = 0, \quad \forall \phi \in V_h^0,$$

where (\cdot, \cdot) denotes the $L^2(\Omega)$ scalar product, $(g, \phi)_h := \int_{\Omega} I_h(g\phi)$ corresponds to the use of the vertex quadrature rule to approximate the integral, and $K_h := \{v \in V_h : v|_{\partial\Omega} = -1\}$.

10.2. Discrete barriers. We can prove the following lemma.

Lemma 10.1. *Suppose that \mathcal{S}_h satisfies assumption (A) or (A_{*}). Let $v, w \in V_h$ with $v \leq w$, and $v(N_i) = w(N_i)$ for some internal node $N_i \in \mathcal{N}_h^0$. Then*

$$(\mathcal{L}_h(v), \phi_i) \geq (\mathcal{L}_h(w), \phi_i)$$

if $h = o(\epsilon)$ and ϵ is sufficiently small.

Proof. Let v_j and w_j , $j = 1, \dots, J$, denote the nodal values of v and w , and L_ψ the Lipschitz constant of ψ restricted to $[-2, 2]$. The equality $v_i = w_i$ implies $I_h[\psi(v)](N_i) = I_h[\psi(w)](N_i)$, whence

$$\begin{aligned} (\mathcal{L}_h(w) - \mathcal{L}_h(v), \phi_i) &= \epsilon^2 \sum_{j \neq i} a_{ij}(w_j - v_j) + \sum_{j \neq i} m_{ij}(\psi(w_j) - \psi(v_j)) \\ &\leq \sum_{j \neq i} [\epsilon^2 a_{ij}(w_j - v_j) + m_{ij} L_\psi(w_j - v_j)] \leq 0 \end{aligned}$$

thanks to property (10.2) and the fact that $h = o(\epsilon)$ (j ranges from 1 to J in all summations). \square

Based on Lemma 10.1, we can now prove the following discrete version of Lemma 5.1.

Lemma 10.2 (Discrete Comparison Lemma). *Assume \mathcal{S}_h satisfies assumption (A) or (A $_*$). Suppose $v^-, v^+ \in K_h$ with $v^- \leq v^+$, and $(\mathcal{L}_h(v^-), \phi) \leq 0$, $(\mathcal{L}_h(v^+), \phi) \geq 0$ for all $0 \leq \phi \in V_h^0$ (i.e., v^- and v^+ are respectively a discrete subsolution and a discrete supersolution). Then there exists $u \in K_h$, $v^- \leq u \leq v^+$, a solution of the discrete problem (10.3).*

Proof. For this proof, which is based on a fixed point argument, we shall identify any function $v \in K_h$ with the vector in \mathbf{R}^I of internal nodal values $v_i = v(N_i)$, $i = 1, \dots, I$.

Introduce $Q \subset \mathbf{R}^I$ to be the closed convex set $Q = \{v \in K_h : v^- \leq v \leq v^+\}$, define the continuous map $F : \mathbf{R}^I \rightarrow \mathbf{R}^I$ as

$$F_i(v) = (\mathcal{L}_h(v), \phi_i), \quad \forall v \in K_h, \quad i = 1, \dots, I,$$

and finally the map $T : Q \rightarrow Q$ by

$$T(v) = P_Q(v - F(v)),$$

where $P_Q : \mathbf{R}^I \rightarrow Q$ is the componentwise projection on Q .

By the Browder fixed point theorem there exists a fixed point $u \in Q$ for the map T . The vector u represents a function (again denoted by u) in K_h satisfying $v^- \leq u \leq v^+$. It remains to prove that u solves problem (10.3), which is equivalent to showing that $F(u) = 0$.

Let $w = u - F(u)$. We claim that $w \in Q$. This would imply the result, since in this case $P_Q(w) = w$, which gives $u = T(u) = w$ and hence $F(u) = 0$.

To see that $w \in Q$, suppose by contradiction that $w_i < v_i^-$ at some node $1 \leq i \leq I$ (the proof is similar if $w_i > v_i^+$). Since $u = P_Q(w)$, we have $u_i = v_i^-$, and we can apply Lemma 10.1 to the functions $v^- \leq u \in K_h$, obtaining

$$(\mathcal{L}_h(v^-), \phi_i) \geq (\mathcal{L}_h(u), \phi_i).$$

This, in conjunction with the fact that v^- is a subsolution implies that $F_i(u) = (\mathcal{L}_h(u), \phi_i) \leq 0$, hence $w_i = u_i - F_i(u) \geq u_i = v_i^-$, which contradicts the assumption on w_i . \square

By using the continuous barriers constructed in §7, by means of the elliptic projection defined in (10.1), we now define the discrete barriers as

$$(10.4) \quad v_{\epsilon, h}^- = \Pi_h(v_\epsilon^-), \quad v_{\epsilon, h}^+ = \Pi_h(v_\epsilon^+).$$

Lemma 10.3 (L^∞ stability and error estimate). *Suppose $h^2 |\log h| = o(\epsilon^2)$; then the discrete barrier $v_{\epsilon, h}^-$ satisfies the following stability estimates:*

$$(10.5) \quad \|v_{\epsilon, h}^-\|_{L^\infty(\Omega)} \leq 2, \quad \|\nabla v_{\epsilon, h}^-\|_{L^\infty(\Omega)} \leq \frac{C}{\epsilon},$$

for sufficiently small ϵ and h . Moreover, the following $L^\infty(\Omega)$ error estimate holds:

$$(10.6) \quad \|v_\epsilon^- - v_{\epsilon, h}^-\|_{L^\infty(\Omega)} \leq Ch^2 |\log h| \|D^2 v_\epsilon^-\|_{L^\infty(\Omega)} \leq C \frac{h^2}{\epsilon^2} |\log h|.$$

Similar estimates are valid for $v_{\epsilon, h}^+$.

Proof. The first inequality in (10.6) has been proved by Nitsche [26] and Scott [35] for the elliptic projection; $\|D^2 v_\epsilon^-\|_{L^\infty(\Omega)} \leq C\epsilon^{-2}$ follows from the construction of v_ϵ^- . We also have the following $W^{1,\infty}$ stability estimate (Rannacher, Scott [34]):

$$\|\nabla v_{\epsilon,h}^-\|_{L^\infty(\Omega)} \leq C\|\nabla v_\epsilon^-\|_{L^\infty(\Omega)},$$

which immediately gives the second stability estimate in (10.5) in view of the construction of the continuous barrier v_ϵ^- . Finally, the first estimate in (10.5) follows from (10.6), the hypothesis on h , and the L^∞ bound $\|v_\epsilon^-\|_{L^\infty(\Omega)} \leq \frac{3}{2}$ clearly valid for ϵ small enough. \square

Using Lemma 10.3, we are in a position to prove that $v_{\epsilon,h}^-$ is a discrete subsolution.

Theorem 10.1. *Under the following restriction on h ,*

$$(10.7) \quad h = \mathcal{O}(\epsilon^{5/2}),$$

the discrete function $v_{\epsilon,h}^-$ is a discrete subsolution, i.e., $(\mathcal{L}_h(v_{\epsilon,h}^-), \phi) \leq 0$ for all $0 \leq \phi \in V_h^0$.

Proof. We decompose $(\mathcal{L}_h(v_{\epsilon,h}^-), \phi)$ as follows:

$$\begin{aligned} (\mathcal{L}_h(v_{\epsilon,h}^-), \phi) &= (\mathcal{L}(v_\epsilon^-), \phi) + \left[(I_h \psi(v_{\epsilon,h}^-), \phi) - (\psi(v_{\epsilon,h}^-), \phi) \right] \\ &\quad + \left[(\psi(v_{\epsilon,h}^-), \phi) - (\psi(v_\epsilon^-), \phi) \right] \\ &\quad + \epsilon \frac{c_0}{2} [(g, \phi) - (g, \phi)_h] =: \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

From §§7.1, 7.2, 7.3, we have $I \leq -C\epsilon^3 |\log \epsilon|^2 \|\phi\|_{L^1(\Omega)}$. Let $L_{\psi'}$ be the Lipschitz constant of ψ' restricted to $[-2, 2]$. Using standard properties of the interpolation operator I_h , in conjunction with (10.5), we can estimate

$$\begin{aligned} \text{II} &\leq \|I_h \psi(v_{\epsilon,h}^-) - \psi(v_{\epsilon,h}^-)\|_{L^\infty(\Omega)} \|\phi\|_{L^1(\Omega)} \leq Ch^2 \max_{T \in \mathcal{S}_h} \|D^2 \psi(v_{\epsilon,h}^-)\|_{L^\infty(T)} \|\phi\|_{L^1(\Omega)} \\ &\leq Ch^2 L_{\psi'} \|\nabla v_{\epsilon,h}^-\|_{L^\infty(\Omega)}^2 \|\phi\|_{L^1(\Omega)} \leq C \frac{h^2}{\epsilon^2} \|\phi\|_{L^1(\Omega)}. \end{aligned}$$

Using (10.6), we have

$$\begin{aligned} \text{III} &\leq \|\psi(v_{\epsilon,h}^-) - \psi(v_\epsilon^-)\|_{L^\infty(\Omega)} \|\phi\|_{L^1(\Omega)} \leq L_\psi \|v_{\epsilon,h}^- - v_\epsilon^-\|_{L^\infty(\Omega)} \|\phi\|_{L^1(\Omega)} \\ &\leq C \frac{h^2}{\epsilon^2} |\log h| \|\phi\|_{L^1(\Omega)}. \end{aligned}$$

Finally, in view of the regularity of g and the inverse inequality $\|\nabla \phi\|_{L^1(T)} \leq \frac{C}{h_T} \|\phi\|_{L^1(T)}$, for any $T \in \mathcal{S}_h$, where h_T is the diameter of T , the quadrature error can be bounded as follows:

$$\begin{aligned} \text{IV} &\leq C\epsilon \sum_{T \in \mathcal{S}_h} h_T^2 \|D^2(g\phi)\|_{L^1(T)} \\ &\leq C\epsilon \sum_{T \in \mathcal{S}_h} h_T^2 [\|\nabla g\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^1(T)} + \|D^2 g\|_{L^\infty(\Omega)} \|\phi\|_{L^1(T)}] \\ &\leq C\epsilon h \|\phi\|_{L^1(\Omega)}. \end{aligned}$$

The terms II and III can be controlled by term I if we require

$$\frac{h^2}{\epsilon^2} |\log h| = \mathcal{O}(\epsilon^3 |\log \epsilon|^2),$$

which is seen true by using (10.7). Term I also controls IV, as can be seen by exploiting (10.7). \square

10.3. The discrete solution. The discrete barriers $v_{\epsilon,h}^-$, $v_{\epsilon,h}^+$ defined in (10.4) cannot be directly employed in Lemma 10.2, since they do not belong to the affine space K_h . We can however simply modify, e.g., $v_{\epsilon,h}^-$ by raising the value on the boundary nodes to be -1 . After this modification $v_{\epsilon,h}^-$ is *a fortiori* a subsolution (apply Lemma 10.1 to the original and the modified $v_{\epsilon,h}^-$).

The requirement $v_{\epsilon,h}^- \leq v_{\epsilon,h}^+$ is a consequence of the fact that $v_{\epsilon}^+ - v_{\epsilon}^- > C\epsilon^3 |\log \epsilon|^2$ (direct verification) by using the error estimate (10.6) and the relation (10.7).

We can finally apply Lemma 10.2 to get a solution $v_{\epsilon,h}^- \leq u_{\epsilon,h} \leq v_{\epsilon,h}^+$ of problem (10.3).

10.4. Discrete interface error estimate. The constructed discrete barriers $v_{\epsilon,h}^-$ and $v_{\epsilon,h}^+$ allow us to prove a discrete equivalent of Theorem 8.1 for the Hausdorff distance between the exact prescribed curvature surface Σ and the discrete surface defined by $\Sigma_{\epsilon,h} := \{u_{\epsilon,h} = 0\}$.

Theorem 10.2. *Under the hypotheses of Theorem 8.1, with g compactly supported in Ω , and under the assumption (A) or (A_{*}) for the mesh \mathcal{S}_h and the relation (10.7) between the relaxation parameter ϵ and the mesh size h , there exist a constant C independent of ϵ and $\epsilon_0 > 0$ such that*

$$\text{dist}_H(\Sigma, \Sigma_{\epsilon,h}) \leq C\epsilon^2 |\log \epsilon|^2, \quad \forall \epsilon < \epsilon_0.$$

Proof. Sticking with the notation of Theorem 8.1, we claim that $A^- \subseteq \{v_{\epsilon,h}^- \geq 0\} \subseteq \{v_{\epsilon,h}^+ \geq 0\} \subseteq A^+$. This will immediately imply the result, thanks to property (8.2) of the Hausdorff distance and the relation $v_{\epsilon,h}^- \leq u_{\epsilon,h} \leq v_{\epsilon,h}^+$.

To prove the claim, simply observe that

$$\begin{aligned} v_{\epsilon,h}^-(x) &\geq v_{\epsilon}^-(x) - \|v_{\epsilon,h}^- - v_{\epsilon}^-\|_{L^\infty(\Omega)} \\ &\geq \gamma_{\epsilon} \left(\frac{d_{\epsilon}^-(x)}{\epsilon} \right) - 2\epsilon \|g\|_{L^\infty(\Omega)} \|\eta\|_{L^\infty(\Omega)} - C \frac{h^2}{\epsilon^2} |\log h|. \\ &\geq \gamma_{\epsilon} \left(\frac{d_{\epsilon}^-(x)}{\epsilon} \right) - C\epsilon, \end{aligned}$$

where we used definition (7.2) of v_{ϵ}^- , estimate (10.6) and assumption (10.7). In view of the definition of A^- with C_1 large enough we get the first inclusion. The second one follows similarly. \square

11. A NUMERICAL SIMULATION

For the sake of completeness we present a numerical simulation included in [5] for the solution of a prescribed curvature problem.

Set $\Omega = (-4, 4) \times (-4, 4)$ and take $g(x) = -|x| + 5/2$; then $A = \{x \in \Omega : |x| < 2\}$ is a nondegenerate absolute minimizer of $\tilde{\mathcal{F}}$ as defined in (3.1).

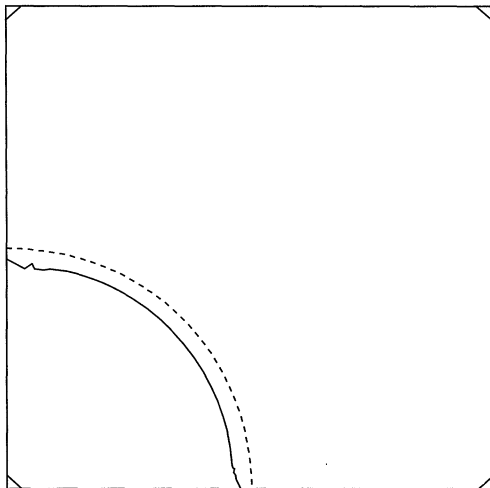


FIGURE 11.1. Exact solution (dashed line) and discrete solution (solid line)

We discretize Ω with a uniform unstructured triangular mesh of size $h = 0.1$, fix $\epsilon = 0.17$ and solve the discrete problem (10.3) with the addition of a mass lumping on the nonlinear term involving ψ (see Conclusions, §12) by using a nonlinear Jacobi iteration scheme. We enforce the symmetry of the problem by solving (10.3) on the quadrant $(0, 4) \times (0, 4)$ with homogeneous Neumann boundary conditions on the coordinate axes. The exact solution ∂A and the computed solution $\Sigma_{\epsilon, h}$ are shown in Figure 11.1.

12. CONCLUSIONS

The boundary $\partial\Omega$ and the choice of the boundary condition for problem (3.2) plays an important role in Theorem 8.1. Other different boundary conditions such as homogeneous Neumann or general Dirichlet conditions could however be considered with only slight changes in the proofs, if we keep requiring the relative compactness of the minimizer $A \subset\subset \Omega$.

It seems not at all trivial to relax this compactness assumption, unless specific choices for the boundary condition (Homogeneous Neumann) and on $\partial\Omega$ (e.g. planar where Σ intersects $\partial\Omega$) are taken. The most interesting case of a general Dirichlet boundary condition, which leads to prescribed contact angle for the minimizer to $\partial\Omega$, seems to require a precise construction of the subsolution in the neighborhood of the contact set $\Sigma \cap \partial\Omega$. Moreover, the definition of *nondegenerate* minimizer has to be reformulated, since now the manifold $\Sigma \cap \Omega$ has a boundary.

The discrete problem formulated in (10.3) does not include mass lumping for the *reaction* term $\psi(v)$, as was done in [4], [5], leading to a nondiagonal mass matrix in the resulting nonlinear system. Indeed, the use of mass lumping can only be taken into account by a stricter restriction $h = \mathcal{O}(\epsilon^4)$, since the difference $V := (I_h \psi(v_{\epsilon, h}^-), \phi)_h - (I_h \psi(v_{\epsilon, h}^-), \phi)$ now arising in the proof of Theorem 10.1 can

be bounded by

$$|V| \leq C \sum_{T \in \mathcal{S}_h} h_T^2 \|\nabla \psi(v_{\epsilon, h}^-)\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^1(T)} \leq C \frac{h}{\epsilon} \|\phi\|_{L^1(\Omega)}$$

and can be controlled by term I if $h = \mathcal{O}(\epsilon^4)$.

REFERENCES

1. S.M. Allen and J.W. Cahn, *A macroscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta Metall. **27** (1979), 1085–1095
2. G. Barles, H.-M. Soner, and P.E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim. **31** (1993), 439–469. MR **94c**:35005
3. G. Bellettini and M. Paolini, *Quasi-optimal error estimates for the mean curvature flow with a forcing term*, Differential Integral Equations **8** (1995), 735–752. MR **95h**:35021
4. G. Bellettini, M. Paolini, and C. Verdi, Γ -convergence of discrete approximations to interfaces with prescribed mean curvature, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **1** (1990), 317–328. MR **92a**:49019
5. ———, *Numerical minimization of geometrical type problems related to calculus of variations*, Calcolo **27** (1990), 251–278. MR **92i**:65103
6. E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268. MR **40**:3445
7. L. Bronsard and R.V. Kohn, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations **90** (1991), 211–237. MR **92d**:35037
8. X. Chen, *Generation and propagation of interfaces in reaction-diffusion equations*, J. Differential Equations **96** (1992), 116–141. MR **92m**:35129
9. X. Chen and C.M. Elliott, *Asymptotics for a parabolic double obstacle problem*, Proc. Roy. Soc. London Ser. A **444** (1994), 429–445. MR **95f**:35128
10. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978. MR **58**:25001
11. R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volume I*, Interscience Publishers Inc., New York, 1953. MR **16**:426a
12. E. De Giorgi, *Congetture sui limiti delle soluzioni di alcune equazioni paraboliche quasi lineari*, Nonlinear Analysis. A Tribute in Honour of G. Prodi, S.N.S. Quaderni, Pisa, 1991, pp. 173–187.
13. E. De Giorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **58** (1975), 842–850. MR **56**:6503
14. P. De Mottoni and M. Schatzman, *Geometrical evolution of developed interfaces*, Trans. Amer. Math. Soc. **347** (1995), 1533–1589.
15. G. Dziuk and J.E. Hutchinson, *On the approximation of unstable parametric minimal surfaces*, Calc. Var. Partial Differential Equations **4** (1996), 27–58. CMP 96:09
16. L.C. Evans, H.-M. Soner, and P.E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. **45** (1992), 1097–1123. MR **93g**:35064
17. H. Federer, *Geometric Measure Theory*, Springer-Verlag, Berlin, 1968. MR **41**:1976
18. R. Finn, *Equilibrium Capillary Surfaces*, Springer-Verlag, Berlin, 1986. MR **88f**:49001
19. A. Friedman, *Variational Principles and Free Boundary Problems*, Wiley, New York, 1982. MR **84e**:35153
20. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983. MR **86c**:35035
21. E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984. MR **87a**:58041
22. S. Helgason, *Groups and Geometric Analysis*, Academic Press, Orlando, Florida, 1984. MR **86c**:22017
23. T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom. **38** (1993), 417–461. MR **94h**:58051
24. H. Lewy, *Aspects of the Calculus of Variations*, University of California Press, Berkeley, 1939
25. L. Modica and S. Mortola, *Un esempio di Γ -convergenza*, Boll. Un. Mat. Ital. B (5) **14** (1977), 285–299. MR **56**:3704

26. J. Nitsche, *L_∞ -convergence of finite element approximations*, Mathematical Aspects of Finite Element Methods, Lecture Notes in Math., 606., Springer-Verlag, Berlin, 1977, pp. 261–274. MR **58**:8351
27. J.C.C. Nitsche, *Lectures on Minimal Surfaces, Volume 1*, Cambridge University Press, Cambridge, 1989. MR **90m**:49031
28. R.H. Nochetto, M. Paolini, and C. Verdi, *Sharp error analysis for curvature dependent evolving fronts*, Math. Models Methods Appl. Sci. **3** (1993), 711–723. MR **94k**:35028
29. ———, *Optimal interface error estimates for the mean curvature flow*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), 193–212. MR **96b**:35020
30. ———, *Quadratic rate of convergence for curvature dependent smooth interfaces: a simple proof*, Appl. Math. Lett. **7** (1994), 59–63. MR **96d**:35067
31. ———, *Double obstacle formulation with variable relaxation parameter for smooth geometric front evolutions: asymptotic interface error estimates*, Asymptotic Anal. **10** (1995), 173–198. MR **96b**:35238
32. ———, *A dynamic mesh algorithm for curvature dependent evolving interfaces*, J. Comput. Phys. **123** (1996), 296–310. CMP 96:07
33. M. Paolini and C. Verdi, *Asymptotic and numerical analyses of the mean curvature flow with a space-dependent relaxation parameter*, Asymptotic Anal. **5** (1992), 553–574. MR **93h**:35101
34. R. Rannacher and R. Scott, *Some optimal error estimates for piecewise linear finite element approximations*, Math. Comp. **38** (1982), 437–445. MR **83e**:65180
35. R. Scott, *Optimal L^∞ estimates for the finite element method on irregular meshes*, Math. Comp. **32** (1978), 681–697. MR **55**:9560
36. J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105. MR **38**:1617
37. J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983. MR **84d**:35002

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI UDINE, 33100, UDINE, ITALY
E-mail address: paolini@dimi.uniud.it